# Non-factorizable contributions in nonleptonic weak interactions of $K$ mesons

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## Abstract

$K \to \pi\pi$, $K_L$-$K_S$ mass difference and $K_L \to \gamma\gamma^{(*)}$ are studied systematically by decomposing their amplitude into a sum of factorizable and non-factorizable ones. The former is calculated by using the naive factorization while the latter is assumed to be dominated by dynamical contributions of various hadron states. Non-factorizable amplitudes for the $K \to \pi\pi$ decays are estimated by using a hard pion approximation in the infinite momentum frame. Long distance non-factorizable contributions to the $K_L$-$K_S$ mass difference is dominated by those of pseudo scalar meson poles and $\pi\pi$ intermediate states. The amplitude for the $K_L \to \gamma\gamma^{(*)}$ is given by a sum of pole contributions of pseudo scalar mesons in the $s$-channel and $K^*$ meson in the crossed channel.

By fitting the results on the $K \to \pi\pi$, $K_L$-$K_S$ mass difference and $K_L \to \gamma\gamma$ to the observations, values of unknown parameters involved are estimated and then, by using the resulting values of these parameters, the form factor for the Dalitz decays of $K_L$ and their rates are predicted. The results are compared with the existing data.

# I. INTRODUCTION

It has been known that short distance contribution is small in the $K_L \to \gamma\gamma$ decay [1] and a naively factorized $|\Delta \mathbf{I}| = \frac{1}{2}$ amplitude for the $K \to \pi\pi$ decays is much smaller than the observed one [2]. Therefore, in these weak processes, it is expected that non-factorizable long distance contribution will play an important role. However, in the $K_L$-$K_S$ mass difference ($\Delta m_K = m_{K_L} - m_{K_S}$), significance of non-factorizable contribution is still not clear. Pseudo scalar meson poles [3] can give a right size of the mass difference but the predicted sign is opposite to the observation (although it is sensitive to the $\eta$-$\eta'$ mixing). Contribution of $\pi\pi$ intermediate states [4] can occupy about a half of the observed mass difference. Short distance contribution has been used to test theories within or beyond the standard model by assuming explicitly or implicitly dominance of factorizable short distance contribution. Therefore, it will be meaningful to study systematically these weak processes and check a role of the non-factorizable long distance contribution in a overall consistent way.

Our starting point to study nonleptonic weak processes is to decompose their amplitude into a sum of *factorizable* and *non-factorizable* ones [5] and therefore, for example, the $|\Delta S| = 1$ effective weak Hamiltonian is divided into the corresponding parts,

$$H_w = (H_w)_{\text{FA}} + (H_w)_{\text{NF}}, \tag{1}$$

where $(H_w)_{\text{FA}}$ and $(H_w)_{\text{NF}}$ are responsible for factorizable and non-factorizable amplitudes, respectively. The factorizable amplitude is estimated by using the naive factorization [6,7]. Then, assuming that the non-factorizable amplitude is dominated by dynamical contributions of various hadrons and using a hard pion approximation in the infinite momentum frame (IMF) [8,9], we estimate the non-factorizable amplitudes for the $K \to \pi\pi$ decays. The hard pion amplitude will be given by *asymptotic* matrix elements of $(H_w)_{\text{NF}}$ (matrix elements of $(H_w)_{\text{NF}}$ taken between single hadron states with infinite momentum).

Before we study explicitly these weak processes, we will investigate constraints on asymptotic matrix elements of $(H_w)_{\text{NF}}$ in the next section and find, using a simple quark counting, that the asymptotic ground-state-meson matrix elements (matrix elements taken between ground-state-meson states with infinite momentum) satisfy the $|\Delta \mathbf{I}| = \frac{1}{2}$ rule. In Sec. III, the $K \to \pi\pi$ decays will be explicitly investigated. Since the naively factorized amplitude does not satisfy the $|\Delta \mathbf{I}| = \frac{1}{2}$ rule and its $|\Delta \mathbf{I}| = \frac{1}{2}$ part is much smaller than the observed one, the non-factorizable part should dominate the amplitude and satisfy the approximate $|\Delta \mathbf{I}| = \frac{1}{2}$ rule to reproduce the observation. To realize this, in the present approach, the asymptotic matrix elements of $(H_w)_{\text{NF}}$ should be much larger than corresponding ones of $(H_w)_{\text{FA}}$ and satisfy the $|\Delta \mathbf{I}| = \frac{1}{2}$ rule as will be shown in Sec. II. The $K_L$-$K_S$ mass difference ($\Delta m_K$) will be investigated, in Sec. IV, by decomposing it into a sum of *short distance* and *long distance* contributions [10],

$$\Delta m_K = (\Delta m_K)_{\text{SD}} + (\Delta m_K)_{\text{LD}}. \tag{2}$$

Its short distance term, $(\Delta m_K)_{\text{SD}}$, is given by a matrix element $\langle K^0|H_{\Delta S=2}|\bar{K}^0\rangle$, where $H_{\Delta S=2}$ is the $|\Delta S| = 2$ effective Hamiltonian arising from the so-called box diagrams [1]. It will be again divided into a sum of factorizable and non-factorizable parts,

$$H_{\Delta S=2} = (H_{\Delta S=2})_{\text{FA}} + (H_{\Delta S=2})_{\text{NF}}. \tag{3}$$

The long distance contribution $(\Delta m_K)_{\mathrm{LD}}$ is given by a sum of the Born term and the continuum contribution [11]. The former will be dominated by contributions of pseudo-scalar meson poles in the present case [12], although contribution of vector meson poles has been considered for long time [13]. The latter is dominated by $(\pi\pi)$ intermediate states. We will use the result on the $(\pi\pi)$ continuum contribution in Ref. [4] as an input data. The contribution of pseudo-scalar meson poles will be given by asymptotic matrix elements of $H_w$. These matrix elements are dominated by non-factorizable ones as will be seen in Sec. III. We will study, in Sec. V, two photon decays, $K_L \to \gamma\gamma^{(*)}$, and the Dalitz decays, $K_L \to \gamma\ell^+\ell^-$, where $\gamma^{(*)}$ and $\ell$ denote an (off-mass-shell) photon and a lepton, respectively. By assuming the vector meson dominance (VMD) [14], the amplitude will be described approximately by two independent matrix elements of $(H_w)_{\mathrm{NF}}$ taken between pseudo scalar meson states and between helicity $\lambda = \pm 1$ vector meson states. Insertion of constraints on the matrix elements of $(H_w)_{\mathrm{NF}}$ (given in Sec. II) into the long distance amplitude makes us enable to compare our result with experimental data on $K \to \pi\pi$, $\Delta m_K$, $K_L \to \gamma\gamma$ and $K_L \to \gamma\ell^+\ell^-$ in Sec. VI. A brief summary will be given in the final section.

## II. ASYMPTOTIC MATRIX ELEMENTS OF THE EFFECTIVE WEAK HAMILTONIANS

We will describe approximately non-factorizable amplitudes for $K \to \pi\pi$ and $K_L \to \gamma\gamma^{(*)}$ decays as well as the $K^0$-$\bar{K}^0$ transition using asymptotic matrix elements of non-factorizable Hamiltonians, $(H_w)_{\mathrm{NF}}$ and $(H_{\Delta S=2})_{\mathrm{NF}}$, which will be given by color singlet sums of colored current products, and then see that the approximate $|\Delta \mathbf{I}| = \frac{1}{2}$ rule in the $K \to \pi\pi$ decays is mainly controlled by the same selection rule in the asymptotic ground-state-meson matrix elements of $(H_w)_{\mathrm{NF}}$. We will see also that $\langle K^0|(H_{\Delta S=2})_{\mathrm{NF}}|\bar{K}^0\rangle$ is related to $\langle \pi^0|(H_w^{\Delta I=3/2})_{\mathrm{NF}}|\bar{K}^0\rangle$ and hence the former should vanish if the ground-state-meson matrix elements of $(H_w)_{\mathrm{NF}}$ satisfy the $|\Delta \mathbf{I}| = \frac{1}{2}$ rule, i.e., $\langle \pi^0|(H_w^{\Delta I=3/2})_{\mathrm{NF}}|\bar{K}^0\rangle = 0$.

Before we study asymptotic matrix elements of $(H_w)_{\mathrm{NF}}$ and $(H_{\Delta S=2})_{\mathrm{NF}}$, we review briefly these effective Hamiltonians. The main part of the $|\Delta S| = 1$ effective weak Hamiltonian is usually written in the form [15,16],

$$H_w = \frac{G_F}{\sqrt{2}} V_{ud} V_{us} \Big\{ c_1 O_1 + c_2 O_2 + (\text{penguin}) \Big\} + h.c., \tag{4}$$

where the four quark operators $O_1$ and $O_2$ are given by products of color singlet left-handed currents,

$$O_1 =: (\bar{u}s)_{V-A}(\bar{d}u)_{V-A} : \quad \text{and} \quad O_2 =: (\bar{u}u)_{V-A}(\bar{d}s)_{V-A} : . \tag{5}$$

$V_{ij}$ denotes a CKM matrix element [17] which is taken to be real since CP invariance is always assumed in this paper.

When we calculate the factorizable amplitudes for the $K \to \pi\pi$ decays later, we use, as usual, the so-called BSW Hamiltonian [7,18]

$$H_w^{BSW} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us} \Big\{ a_1 O_1 + a_2 O_2 \Big\} + h.c. \tag{6}$$

which can be obtained from Eq.(4) by using the Fierz reordering. The operators $O_1$ and $O_2$ in Eq.(6) should be no longer Fierz reordered. We now replace $(H_w)_{\mathrm{FA}}$ by $H_w^{BSW}$ as usual. The penguin term has been neglected in $H_w^{BSW}$ since its contribution to the naively factorized amplitude is expected to be very small [2] in contrast with the old expectation [16]. The coefficients $a_1$ and $a_2$ are given by

$$a_1 = c_1 + \frac{c_2}{N_c} \simeq 1.14, \quad a_2 = c_2 + \frac{c_1}{N_c} \simeq -0.209, \tag{7}$$

where $N_c$ is the color degree of freedom. Their numerical values have been given by using the values of $c_1$ and $c_2$ with the leading order QCD corrections [19].

When $H_w^{BSW}$ is obtained, an extra term which is given by a color singlet sum of products of colored currents,

$$\tilde{H}_w = \frac{G_F}{\sqrt{2}} V_{ud} V_{us} \Big\{ c_2 \tilde{O}_1 + c_1 \tilde{O}_2 + (\text{penguin}) \Big\} + \text{h.c.}, \tag{8}$$

comes out, where

$$\tilde{O}_1 = 2 \sum_a : (\bar{u} t^a s)_{V-A} (\bar{d} t^a u)_{V-A} : \quad \text{and} \quad \tilde{O}_2 = 2 \sum_a : (\bar{u} t^a u)_{V-A} (\bar{d} t^a s)_{V-A} : \tag{9}$$

with the generators $t^a$ of the color $SU_c(N_c)$ symmetry. Eq.(8) can be rewritten in the form,

$$\tilde{H}_w = \frac{G_F}{\sqrt{2}} V_{ud} V_{us} \Big\{ \tilde{c}_- \tilde{O}_- + \tilde{c}_+ \tilde{O}_+ + (\text{penguin}) \Big\} + h.c. \tag{10}$$

where $\tilde{O}_\pm = \tilde{O}_1 \pm \tilde{O}_2$. $\tilde{O}_-$ transforms like $\mathbf{8_a}$ (not $\mathbf{8_s}$) and $\tilde{O}_+$ includes a component transforming like $\mathbf{27}$ of the flavor $SU_f(3)$. They are responsible for the non-factorizable $|\Delta \mathbf{I}| = \frac{1}{2}$ and $\frac{3}{2}$ amplitudes for the $K \to \pi\pi$ decays, respectively.

The penguin operators in the SVZ scheme [16],

$$O_5 = \sum_a : (\bar{d} t^a s)_{V-A} \sum_q (\bar{q} t^a q)_{V+A} : \quad \text{and} \quad O_6 =: (\bar{d} s)_{V-A} \sum_q (\bar{q} q)_{V+A} : \tag{11}$$

consist of products of left- and right-handed currents so that their contributions to matrix elements taken between two pseudo scalar meson states may be enhanced relatively to those of $\tilde{O}_1$ and $\tilde{O}_2$ which are given by products of left-handed currents. Therefore the penguin term cannot necessarily be neglected in $\tilde{H}_w$. Since $O_6$ consists of products of color singlet currents, it may be factorizable and included in $H_w^{BSW}$. However, the penguin contributions in the factorized amplitudes have been small as mentioned before so that $O_5$ which consists of colored current products survives in $\tilde{H}_w$. To realize a physical process described in terms of a matrix element, $\langle P_2 P_3 | \tilde{H}_w | P_1 \rangle$, soft gluon(s) have to be exchanged between quark(s) and anti-quark(s) belonging to different meson states since $\tilde{H}_w$ is given by a color singlet sum of colored current products as seen the above. Therefore, $\langle P_2 P_3 | \tilde{H}_w | P_1 \rangle$ is not factorizable, i.e., $(H_w)_{\mathrm{NF}}$ is given by $\tilde{H}_w$. In this way, the $|\Delta S| = 1$ effective weak Hamiltonian is now given in the form,

$$H_w \to H_w^{BSW} + \tilde{H}_w. \tag{12}$$

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In the same way, the $\Delta S = 2$ effective Hamiltonian,

$$H_{\Delta S=2} = c_{\Delta S=2} O_{\Delta S=2} \; + \; h.c \quad \text{with} \quad O_{\Delta S=2} =: (\bar{d}s)_{V-A}(\bar{d}s)_{V-A} :, \tag{13}$$

is again replaced by a sum of the Fierz reordered BSW-like Hamiltonian, $H^{BSW}_{\Delta S=2}$, and an extra term, $\tilde{H}_{\Delta S=2}$, i.e.,

$$H_{\Delta S=2} \to H^{BSW}_{\Delta S=2} + \tilde{H}_{\Delta S=2}, \tag{14}$$

where $H^{BSW}_{\Delta S=2}$ is obtained by applying the Fierz reordering to the above $H_{\Delta S=2}$. The extra term is again given by a color singlet sum of products of colored currents,

$$\tilde{H}_{\Delta S=2} = c_{\Delta S=2}\tilde{O}_{\Delta S=2} \; + \; h.c. \quad \text{with} \quad \tilde{O}_{\Delta S=2} = 2\sum_a : (\bar{d}t^a s)_{V-A}(\bar{d}t^a s)_{V-A} : . \tag{15}$$

Now we study constraints on matrix elements of non-factorizable Hamiltonians [20,21]. The non-factorizable four-quark operators $\tilde{O}_\pm$ and $\tilde{O}_{\Delta S=2}$ can be expanded into a sum of products of (a) two creation operators to the left and two annihilation operators to the right, (b) three creation operators to the left and one annihilation operator to the right, (c) one creation operator to the left and three annihilation operators to the right, and (d) all (four) creation operators or annihilation operators of quarks and anti-quarks. We associate (a)−(d) with quark-line diagrams describing different types of matrix elements of $\tilde{O}_\pm$ and $\tilde{O}_{\Delta S=2}$. For (a), we utilize the two creation and annihilation operators to create and annihilate, respectively, the quarks and anti-quarks belonging to the meson states $\langle\{q\bar{q}\}|$ and $|\{q\bar{q}\}\rangle$ in the asymptotic matrix elements of the four-quark operators, $\tilde{O}_\pm$ and $\tilde{O}_{\Delta S=2}$. For (b) and (c), we need to add a spectator quark or anti-quark to reach *physical* processes, for example, $\langle\{qq\bar{q}\bar{q}\}|\tilde{O}_\pm|\{q\bar{q}\}\rangle$ and $\langle\{q\bar{q}\}|\tilde{O}_\pm|\{qq\bar{q}\bar{q}\}\rangle$. Here $\{qq\bar{q}\bar{q}\}$ denotes a four-quark meson [22].

While we count all possible connected quark-line diagrams, we forget color degree of freedom of quarks since they will be compensated by a deep sea of soft gluons carried by light mesons. However, we have to be careful with the order of the quark(s) and anti-quark(s) in $\tilde{O}_\pm$ and $\tilde{O}_{\Delta S=2}$ since symmetry (or antisymmetry) property of wave functions of meson states under exchanges of quark and anti-quark plays an important role when asymptotic matrix elements of $\tilde{O}_\pm$ and $\tilde{O}_{\Delta S=2}$ are considered. Noting that the wave function of the ground-state $\{q\bar{q}\}_0$ meson is antisymmetric under exchange of its quark and anti-quark [23], we obtain [24],

$$\langle\{q\bar{q}\}_0|\tilde{O}_+|\{q\bar{q}\}_0\rangle = 0, \tag{16}$$

which implies that the asymptotic ground-state-meson matrix elements of $\tilde{H}_w$ satisfy the $|\Delta \mathbf{I}| = \frac{1}{2}$ rule since the penguin term satisfies it always. We will neglect contributions of excited $\{q\bar{q}\}$ meson states as will be discussed in the next section. The same quark counting leads directly to [25,26]

$$\langle K^0|\tilde{O}_{\Delta S=2}|\bar{K}^0\rangle = 0 \tag{17}$$

which is compatible with Eq.(16).

According to Ref. [22], four quark $\{qq\bar{q}\bar{q}\}$ mesons are classified into the following four types, $\{qq\bar{q}\bar{q}\} = [qq][\bar{q}\bar{q}] \oplus (qq)(\bar{q}\bar{q}) \oplus \{[qq](\bar{q}\bar{q}) \pm (qq)[\bar{q}\bar{q}]\}$, where () and [] denote symmetry

and antisymmetry, respectively, under the exchange of flavors between them. The first two can have $J^P = 0^+$ but the last two have $J^P = 1^+$. Each multiplet classified above is again classified into two different classes according to different combinations of color degree of freedom. However, they can mix with each other. The masses of four-quark mesons with the mixing have been predicted by using the bag model. The members of the heavier class have been expected to play an important role in charm decays [21] since their predicted masses are close to the parent charm meson masses while four-quark meson contribution to $K$ decays will be small since their masses are considerably larger than the kaon mass $m_K$. However, in the $K^+ \to \pi^+ \pi^0$ decay, their contributions are not necessarily negligible since the other contributions are suppressed because of the above $|\Delta \mathbf{I}| = \frac{1}{2}$ rule in the asymptotic ground-state-meson matrix elements of $\tilde{H}_w$, etc. We here take contributions of the $[qq][\bar{q}\bar{q}]$ and $(qq)(\bar{q}\bar{q})$ with $J^{P(C)} = 0^{+(+)}$ belonging to the lower mass class, although contributions of heavier ones may not be negligible for more precise discussions.

Using the same prescription as the above, we can obtain the following constraints on asymptotic matrix elements of $\tilde{H}_w$ between $\{q\bar{q}\}_0$ and $\{qq\bar{q}\bar{q}\}$ meson states [24],

$$\langle [qq][\bar{q}\bar{q}] | \tilde{O}_+ | \{q\bar{q}\}_0 \rangle = \langle \{q\bar{q}\}_0 | \tilde{O}_+ | [qq][\bar{q}\bar{q}] \rangle = 0, \tag{18}$$

$$\langle (qq)(\bar{q}\bar{q}) | \tilde{O}_- | \{q\bar{q}\}_0 \rangle = \langle \{q\bar{q}\}_0 | \tilde{O}_- | (qq)(\bar{q}\bar{q}) \rangle = 0. \tag{19}$$

The above equations imply that asymptotic matrix elements of $\tilde{H}_w$ between $\{q\bar{q}\}_0$ and $[qq][\bar{q}\bar{q}]$ meson states satisfy the $|\Delta \mathbf{I}| = \frac{1}{2}$ rule while the matrix elements between $\{q\bar{q}\}_0$ and $(qq)(\bar{q}\bar{q})$ can violate the rule. Therefore the $|\Delta \mathbf{I}| = \frac{1}{2}$ rule violating non-factorizable amplitude for the $K \to \pi\pi$ decays can be supplied through the $(qq)(\bar{q}\bar{q})$ meson pole amplitudes which can interfere destructively with the too big $|\Delta \mathbf{I}| = \frac{3}{2}$ part of the factorized amplitudes in Table I.

The ground-state-meson matrix elements of $O_5$ is also treated in the same way. Since the flavor $SU_f(3)$ singlet $q\bar{q}$ pairs in the $s$-channel can couple to a glue(-ball) state and induce a matrix element $\langle g_0 | O_5 | \bar{K}^0 \rangle$, where $g_0$ denotes a glue-ball, we treat separately with such contributions when we consider matrix elements $\langle P | O_5 | \bar{K}^0 \rangle$ with $P = \pi^0$, $\eta$, $\eta'$, ... In this way, we obtain

$$\langle \pi^- | O_5 | K^- \rangle = -\sqrt{2} \langle \pi^0 | O_5 | \bar{K}^0 \rangle = \sqrt{2} \langle \eta_0 | O_5 | \bar{K}^0 \rangle = \langle \eta_s | O_5 | \bar{K}^0 \rangle, \tag{20}$$

where $\eta_0 \sim (u\bar{u} + d\bar{d})/\sqrt{2}$ and $\eta_s \sim (s\bar{s})$ are components of iso-singlet pseudo scalar mesons, $\eta$ and $\eta'$,

$$\begin{cases} \eta = a_\eta^0 \eta_0 + a_\eta^s \eta_s = \left( \sqrt{\frac{1}{3}} \cos\theta_P - \sqrt{\frac{2}{3}} \sin\theta_P \right) \eta_0 - \left( \sqrt{\frac{2}{3}} \cos\theta_P + \sqrt{\frac{1}{3}} \sin\theta_P \right) \eta_s, \\ \eta' = a_{\eta'}^0 \eta_0 + a_{\eta'}^s \eta_s = \left( \sqrt{\frac{1}{3}} \sin\theta_P + \sqrt{\frac{2}{3}} \cos\theta_P \right) \eta_0 + \left( -\sqrt{\frac{2}{3}} \sin\theta_P + \sqrt{\frac{1}{3}} \cos\theta_P \right) \eta_s. \end{cases} \tag{21}$$

The mixing angle is usually taken to be $\theta_P \simeq -20°$ [27].

Now we are ready to parameterize the asymptotic ground-state-meson matrix elements of $\tilde{H}_w$. To reproduce the observed $|\Delta \mathbf{I}| = \frac{1}{2}$ rule in the $K \to \pi\pi$ decays, we need the $|\Delta \mathbf{I}| = \frac{1}{2}$ rule for the ground-state-meson matrix elements of $\tilde{H}_w$ with a sufficient precision. It is all right if one accepts the above quark counting. (If not, one should assume the $|\Delta \mathbf{I}| = \frac{1}{2}$ rule for the ground-state-meson matrix elements of $\tilde{H}_w$.) Anyway, neglecting seemingly small (or *zero* in the above quark counting) $|\Delta \mathbf{I}| = \frac{3}{2}$ contributions, we parameterize the (asymptotic)

ground-state-meson matrix elements of $\tilde{H}_w$ as follows,

(A) helicity $\lambda = 0$ matrix elements:

$$
\begin{cases}
\langle \pi^- |\tilde{H}_w|K^-\rangle = \quad (1+r_0)H_0, & \langle \pi^0 |\tilde{H}_w|\bar{K}^0\rangle = -\sqrt{\tfrac{1}{2}}(1+\tilde{r}_0)\tilde{H}_0, \\
\langle \eta_0 |\tilde{H}_w|\bar{K}^0\rangle = -\sqrt{\tfrac{1}{2}}(1-\tilde{r}_0)\tilde{H}_0, & \langle \eta_s |\tilde{H}_w|\bar{K}^0\rangle = \quad \tilde{r}_0\,\tilde{H}_0,
\end{cases}
\tag{22}
$$

(B) helicity $|\lambda| = 1$ matrix elements:

$$
\begin{cases}
\langle \rho^0 |\tilde{H}_w|\bar{K}^{*0}\rangle_1 = -\sqrt{\tfrac{1}{2}}(1+\tilde{r}_1)\tilde{H}_1, & \langle \omega |\tilde{H}_w|\bar{K}^{*0}\rangle_1 = -\sqrt{\tfrac{1}{2}}(1-\tilde{r}_1)\tilde{H}_1, \\
\langle \phi |\tilde{H}_w|\bar{K}^{*0}\rangle_1 = \quad \tilde{r}_1\,\tilde{H}_1.
\end{cases}
\tag{23}
$$

The $\omega$-$\phi$ mixing has been assumed to be ideal. The parameters $\tilde{r}_0$ and $\tilde{r}_1$ denote contributions of the penguin relative to $\tilde{O}_-$ in the helicity $\lambda = 0$ and $|\lambda| = 1$ matrix elements of $\tilde{H}_w$, respectively. $\tilde{H}_0$ and $\tilde{H}_1$ provide their normalizations. In (B), $\tilde{r}_1$ will be neglected hereafter since it is expected to be small because of the small coefficient of the penguin and because of a helicity consideration.

Constraints on asymptotic matrix elements of $\tilde{H}_w$ from Eqs.(18) and (19) including four-quark meson states and their parameterizations will be given in Appendix A.

### III. TWO PION DECAYS OF $K$ MESONS

Now we study two pion decays of $K$ mesons. As mentioned before, an amplitude for $K \to \pi\pi$ decay is given by a sum of *factorizable* and *non-factorizable* ones. The former is estimated by using the naive factorization below while the latter is assumed to be dominated by dynamical contributions of various hadron states and will be estimated later by using a hard pion approximation in the IMF.

The factorizable amplitudes for the $K \to \pi\pi$ decays are estimated by using the naive factorization in the BSW scheme [7]. As an example, we consider the amplitude for the $K^+ \to \pi^+\pi^0$ decay. It is given by

$$
\begin{aligned}
M_{\mathrm{FA}}(K^+ \to \pi^0\pi^+) &= \langle \pi^+(q)\pi^0(p')|H_w^{BSW}|K^+(p)\rangle \\
&= \frac{G_F}{\sqrt{2}}V_{us}V_{ud}\Big\{ a_1\langle \pi^+(q)|(\bar{u}d)_{V-A}|0\rangle\langle \pi^0(p')|(\bar{s}u)_{V-A}|K^+(p)\rangle \\
&\qquad + a_2\langle \pi^0(p')|(\bar{u}u)_{V-A}|0\rangle\langle \pi^+(q)|(\bar{s}d)_{V-A}|K^+(p)\rangle\Big\}.
\end{aligned}
\tag{24}
$$

Factorizable amplitudes for the other $K \to \pi\pi$ decays also can be calculated in the same way. To evaluate these amplitudes, we use the following parameterization of matrix elements of currents,

$$
\begin{cases}
\langle \pi(q)|A_\mu|0\rangle = -if_\pi q_\mu, \quad \text{etc.}, \\
\langle \pi(p')|V_\mu|K(p)\rangle = (p+p')_\mu f_+^{(\pi K)}(q^2) + q_\mu f_-^{(\pi K)}(q^2), \quad \text{etc.},
\end{cases}
\tag{25}
$$

where $q = p-p'$. Using these expressions of current matrix elements, we obtain the factorized amplitudes listed in Table I, where terms proportional to $f_-(q^2)$ have been neglected since their coefficients are small in the spectator decays and, in possible annihilation decays, they

Table I. Naively factorized amplitudes for the $K \to \pi\pi$ decays, where terms proportional to $f_-$ are neglected.

| Decay | $M_{\mathrm{FA}}$ |
|---|---|
| $K_S \to \pi^+\pi^-$ | $iV_{ud}V_{us}(\dfrac{G_F}{\sqrt{2}})\sqrt{2}a_1 f_\pi(m_K^2 - m_\pi^2)f_+^{\pi K}(m_\pi^2)$ |
| $K_S \to \pi^0\pi^0$ | $-iV_{ud}V_{us}(\dfrac{G_F}{\sqrt{2}})a_2 f_\pi(m_K^2 - m_\pi^2)f_+^{\pi K}(m_\pi^2)$ |
| $K^+ \to \pi^+\pi^0$ | $iV_{ud}V_{us}(\dfrac{G_F}{2})(a_1 + a_2)f_\pi(m_K^2 - m_\pi^2)f_+^{\pi K}(m_\pi^2)$ |

are proportional to the small coefficient $a_2$. The naively factorized penguin contribution has also been neglected as discussed in Sec. II. If the values of $a_1$ and $a_2$ with the leading order QCD corrections [2] are taken, it will be seen, since $|a_1| \gg |a_2|$, that the factorized amplitude for the $K^0 \to \pi^0\pi^0$ decay which is described by the color mismatched diagram, $\bar{s} \to \bar{d} + (u\bar{u})_1$, is proportional to $a_2$ and therefore is much smaller (the color suppression) than those for the spectator decays, where $(u\bar{u})_1$ denotes a color singlet pair of $u$ and $\bar{u}$. It is also seen that the factorized amplitude for the $K^+ \to \pi^+\pi^0$ decay is considerably larger than the observed one and that the size of the $|\Delta \mathbf{I}| = \frac{1}{2}$ amplitude is not much larger than the $|\Delta \mathbf{I}| = \frac{3}{2}$ part. Therefore it is hard to reproduce the well-known approximate $|\Delta \mathbf{I}| = \frac{1}{2}$ rule by the naively factorized amplitudes for the $K \to \pi\pi$ decays.

Next we study non-factorizable amplitudes for these decays using a hard pion approximation in the IMF [8,9], i.e., we evaluate the amplitudes at a slightly unphysical point $\mathbf{q} \to 0$ in the IMF ($\mathbf{p} \to \infty$) in which $q_0 \to O(|\mathbf{p}|^{-1})$ and hence $(p \cdot q)$ is still finite. Then the hard pion amplitude as the non-factorizable one is written in the form,

$$M_{\mathrm{NF}}(K \to \pi_1\pi_2) \simeq M_{\mathrm{ETC}}(K \to \pi_1\pi_2) + M_{\mathrm{surf}}(K \to \pi_1\pi_2), \tag{26}$$

where $M_{\mathrm{ETC}}$ and $M_{\mathrm{surf}}$ are given by

$$M_{\mathrm{ETC}}(K \to \pi_1\pi_2) = \frac{i}{\sqrt{2}f_\pi}\langle\pi_2|[V_{\bar{\pi}_1}, \tilde{H}_w]|K\rangle + (\pi_2 \leftrightarrow \pi_1) \tag{27}$$

and

$$M_{\mathrm{surf}}(K \to \pi_1\pi_2) = \frac{i}{\sqrt{2}f_\pi}\Bigg\{\sum_n\Big(\frac{m_\pi^2 - m_K^2}{m_n^2 - m_K^2}\Big)\langle\pi_2|A_{\bar{\pi}_1}|n\rangle\langle n|\tilde{H}_w|K\rangle$$
$$+ \sum_l\Big(\frac{m_\pi^2 - m_K^2}{m_l^2 - m_\pi^2}\Big)\langle\pi_2|\tilde{H}_w|l\rangle\langle l|A_{\bar{\pi}_1}|K\rangle\Bigg\} + (\pi_2 \leftrightarrow \pi_1), \tag{28}$$

respectively, where $[V_\pi + A_\pi, \tilde{H}_w] = 0$ has been used [28] since $\tilde{O}_\pm$ consist of left-handed currents and the right-handed currents in the penguin term is of flavor singlet. The equal-time commutator term, $M_{\mathrm{ETC}}$, has the same form as the one in the old soft pion approximation [29] but now has to be evaluated in the IMF. The surface term, $M_{\mathrm{surf}}$, is given by a

divergent of matrix element of $T$-product of axial vector current and $\tilde{H}_w$ taken between $\langle\pi|$ and $|K\rangle$. However, in contrast with the soft pion approximation, contributions of single meson intermediate states can now survive, when complete sets of energy eigen states are inserted between these two operators. (See, for more details, Refs. [8] and [9].) Therefore, $M_{\text{surf}}$ is given by a sum of all possible pole amplitudes, i.e., $n$ and $l$ in Eq.(28) run over all possible single meson states, not only ordinary $\{q\bar{q}\}$, but also hybrid $\{q\bar{q}g\}$, four-quark $\{qq\bar{q}\bar{q}\}$, glue-balls, etc. However, values of wave functions of orbitally excited $\{q\bar{q}\}_{L\neq0}$ states at the origin are expected to vanish in the non-relativistic quark model and, more generally, wave function overlappings between the ground-state $\{q\bar{q}\}_0$ and excited-state-meson states are expected to be small so that excited-state-meson contributions are not very important except for the non-factorizable $K^+ \to \pi^+\pi^0$ amplitude in which the ground-state-meson contributions can be strongly suppressed because of the $|\Delta\mathbf{I}| = \frac{1}{2}$ rule in the asymptotic ground-state-meson matrix elements of $\tilde{H}_w$ as seen in Sec. II. Asymptotic matrix elements of isospin $V_\pi$ and its axial counterpart $A_\pi$ involved in the amplitudes can be well parameterized by using (asymptotic) $SU_f(3)$ symmetry. Therefore the hard pion amplitude in Eq.(26) with Eqs.(27) and (28) as the non-factorizable long distance contribution is approximately described in terms of asymptotic ground-state-meson matrix elements of $\tilde{H}_w$.

Amplitudes for dynamical hadronic processes, in general, can be described in the form,

$$(\text{continuum contribution}) + (\text{Born term}). \tag{29}$$

In the present case, $M_{\text{surf}}$ is given by a sum of pole amplitudes so that $M_{\text{ETC}}$ corresponds to the continuum contribution [30] which can develop a phase relative to the Born term. Therefore, using isospin eigen amplitudes, $M_{ETC}^{(I)}$'s, and their phases, $\delta_I$'s, we here parameterize the ETC terms as

$$
\begin{cases}
M_{\text{ETC}}(K_S^0 \to \pi^+\pi^-) = \dfrac{2}{3} M_{\text{ETC}}^{(2)}(K \to \pi\pi)e^{i\delta_2} + \dfrac{1}{3} M_{\text{ETC}}^{(0)}(K \to \pi\pi)e^{i\delta_0}, \\[2mm]
M_{\text{ETC}}(K_S^0 \to \pi^0\pi^0) = -\dfrac{2\sqrt{2}}{3} M_{\text{ETC}}^{(2)}(K \to \pi\pi)e^{i\delta_2} + \sqrt{\dfrac{1}{2}} M_{\text{ETC}}^{(0)}(K \to \pi\pi)e^{i\delta_0}, \\[2mm]
M_{\text{ETC}}(K^+ \to \pi^+\pi^0) = M_{\text{ETC}}^{(2)}(K \to \pi\pi)e^{i\delta_2},
\end{cases}
\tag{30}
$$

since the $S$-wave $\pi\pi$ final states can have isospin $I = 0$ and 2. The so-called final state interactions are given by dynamics of hadrons and therefore now they are included in the non-factorizable long distance amplitudes.

By neglecting small contributions of excited states and $|\Delta\mathbf{I}| = \frac{3}{2}$ asymptotic ground-state-meson matrix elements of $\tilde{H}_w$ which vanish in the quark counting as seen before, the non-factorizable amplitudes for the $K \to \pi\pi$ decays can be approximately given as follows,

$$
\begin{cases}
M_{\text{NF}}(K_S^0 \to \pi^+\pi^-) \simeq -\dfrac{i}{f_\pi}\langle\pi^+|\tilde{H}_w|K^+\rangle e^{i\delta_0}, \\[2mm]
M_{\text{NF}}(K_S^0 \to \pi^0\pi^0) \simeq -\sqrt{\dfrac{1}{2}} M_{\text{NF}}(K_S \to \pi^+\pi^-), \\[2mm]
M_{\text{NF}}(K^+ \to \pi^+\pi^0) \simeq 0,
\end{cases}
\tag{31}
$$

where $K^*$ meson pole contributions in the $u$-channel has been neglected [12] although they have been accepted for long time [31]. It is seen that the non-factorizable amplitudes for

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$K \to \pi\pi$ decays have been described approximately in terms of the asymptotic ground-state-meson matrix elements of $\tilde{H}_w$ and the iso-scalar $S$-wave $\pi\pi$ phase shift $\delta_0$ and that they satisfy the $|\Delta \mathbf{I}| = \frac{1}{2}$ rule. More accurate amplitudes involving excited (four-quark) meson contributions will be given in Appendix B. In the last equation of Eq.(31), the right-hand-side is vanishing since the four-quark meson contributions have been neglected. As seen in Appendix B, the $|\Delta \mathbf{I}| = \frac{3}{2}$ part of the non-factorizable amplitude can be supplied through contributions of four-quark meson poles since matrix elements of $\tilde{H}_w$ taken between four-quark $(qq)(\bar{q}\bar{q})$ and the ground-state $\{q\bar{q}\}_0$ meson states can violate the $|\Delta \mathbf{I}| = \frac{1}{2}$ rule as seen in Appendix A. They can interfere destructively with the too big $|\Delta \mathbf{I}| = \frac{3}{2}$ part of the factorized amplitudes in Table I.

## IV. $K_L$-$K_S$ MASS DIFFERENCE

Now we study the $K_L$-$K_S$ mass difference, $\Delta m_K$, by decomposing it into a sum of *short distance* and *long distance* contributions,

$$\Delta m_K = (\Delta m_K)_{\text{SD}} + (\Delta m_K)_{\text{LD}}, \tag{32}$$

as usual [32]. The short distance contribution $(\Delta m_K)_{\text{SD}}$ is given by the matrix element of the $\Delta S = 2$ box operator [1] taken between $\langle K^0|$ and $|\bar{K}^0\rangle$, *i.e.*,

$$(\Delta m_K)_{\text{SD}} = \frac{\langle K^0|H_{\Delta S=2}|\bar{K}^0\rangle}{m_K}. \tag{33}$$

As was seen in Sec. II, it is decomposed into a sum of factorizable and non-factorizable parts,

$$(\Delta m_K)_{\text{SD}} = \{(\Delta m_K)_{\text{SD}}\}_{\text{FA}} + \{(\Delta m_K)_{\text{SD}}\}_{\text{NF}}. \tag{34}$$

The first term on the right-hand-side is estimated by factorizing $\langle K^0|H^{BSW}_{\Delta S=2}|\bar{K}^0\rangle$,

$$\{(\Delta m_K)_{\text{SD}}\}_{\text{FA}} = c_{\Delta S=2}\left\{\frac{8}{3}m_K f_K^2\right\}B_K(\mu), \tag{35}$$

where $B_K(\mu)$ describes the renormalization scale $\mu$ dependence of the four-quark operator $O_{\Delta S=2}$. It compensates the $\mu$ dependence of the corresponding Wilson coefficient, which is given by [33]

$$c_{\Delta S=2} \simeq \frac{G_F^2}{16\pi^2}m_W^2\left\{\lambda_c^2\eta_1 x_c + \lambda_t^2\eta_2 S(x_t) + 2\lambda_c\lambda_t\eta_3 S(x_c, x_t)\right\}b(\mu), \tag{36}$$

where $m_W$ is the weak boson mass and the $\mu$ dependence of $\eta_i$, $(i = 1, 2$ and $3)$, has been factored out by $b(\mu)$. $S(x)$ and $S(x, y)$ are the Inami-Lim functions [34] and $x_i = m_i^2/m_W^2$, $(i = c$ and $t)$. The so-called $B$ factor, $B_K$, is now given by [2]

$$B_K = b(\mu)B_K(\mu). \tag{37}$$

The non-factorizable one is proportional to a matrix element of $\tilde{O}_{\Delta S=2}$ taken between $\langle K^0|$ and $|\bar{K}^0\rangle$. It can be related to the matrix element of the $|\Delta \mathbf{I}| = \frac{3}{2}$ operator, $\tilde{O}_{\Delta I=3/2}$, in the $|\Delta S| = 1$ effective weak Hamiltonian,

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$$\langle K^0|\tilde{O}_{\Delta S=2}|\bar{K}^0\rangle = \sqrt{2}\langle \pi^0|\tilde{O}_{\Delta I=3/2}|\bar{K}^0\rangle, \tag{38}$$

in the $SU_f(3)$ symmetry limit [35] or by using the asymptotic $SU_f(3)$ symmetry [26] which implies [36] a flavor $SU_f(3)$ symmetry of matrix elements of operators (like charges, currents, etc.) taken between single hadron states with **1-8** mixing in the IMF. As was seen before, our quark counting leads to the $|\Delta \mathbf{I}| = \frac{1}{2}$ rule for the asymptotic ground-state-meson matrix elements of $\tilde{H}_w$. It means that the right-hand-side of the above equation vanishes so that the left-hand side also vanishes, i.e., $\langle K^0|\tilde{O}_{\Delta S=2}|\bar{K}^0\rangle = 0$. The same quark counting leads directly to the same result as seen before, and hence we have

$$\{(\Delta m_K)_{\text{SD}}\}_{\text{NF}} = 0. \tag{39}$$

Therefore we do not need to worry about non-factorizable contribution to $(\Delta m_K)_{\text{SD}}$.

The long distance term, $(\Delta m_K)_{\text{LD}}$, has been given by

$$(\Delta m_K)_{\text{LD}} = \int \frac{dm_n^2}{2m_K(m_K^2 - m_n^2)}\Big\{|\langle n|H_w|K_L\rangle|^2 - |\langle n|H_w|K_S\rangle|^2\Big\} \tag{40}$$

in the IMF [11], where the states $n$ consist of a complete set of energy eigen states of hadrons. $(\Delta m_K)_{\text{LD}}$ will be dominated by contributions of pseudo-scalar meson poles and $(\pi\pi)$ intermediate states as discussed in Sec. I,

$$(\Delta m_K)_{\text{LD}} \simeq (\Delta m_K)_{\text{pole}} + (\Delta m_K)_{\pi\pi}. \tag{41}$$

For the $\pi\pi$ continuum contribution, we here take the following value [4],

$$\frac{(\Delta m_K)_{\pi\pi}}{\Gamma_{K_S}} = 0.22 \pm 0.03, \tag{42}$$

which has been obtained by using Omnes-Mushkevili equation and the measured $\pi\pi$ phase shifts, where $\Gamma_{K_S}$ is the full width of $K_S$. Therefore, we hereafter can concentrate on the pole contribution. Since the asymptotic ground-state-meson matrix elements of $H_w$ should be dominated by those of non-factorizable $\tilde{H}_w$ as will be seen in Sec. VI, $(\Delta m_K)_{\text{pole}}$ is given approximately by

$$(\Delta m_K)_{\text{pole}} \simeq \sum_{P_i} \frac{|\langle K_L|\tilde{H}_w|P_i\rangle|^2}{2m_K(M_K^2 - m_{P_i}^2)}, \tag{43}$$

in the IMF, where $P_i = \pi^0$, $\eta$ and $\eta'$. (The $\iota$ contribution has been neglected since it is expected to be not very important because of its high mass.) As seen in Eq.(43), the pole contribution to the $K_L$-$K_S$ mass difference, $(\Delta m_K)_{\text{pole}}$, has been described in terms of asymptotic ground-state-meson matrix elements of $\tilde{H}_w$.

## V. TWO PHOTON DECAYS OF $K_L$

Now we study two photon decays in the IMF for later convenience. The amplitude is given by a sum of short distance and long distance ones. As mentioned before, it is known [1]

that the short distance contribution to the $K_L \to \gamma\gamma$ which is given by the triangle diagram is small. Therefore, we neglect it and consider only long distance effects. The long distance amplitude is again decomposed into a sum of factorizable and non-factorizable ones. The former is given by factorizing the matrix element of $H_w^{BSW}$ taken between $\langle\gamma\gamma|$ and $|K_L^0\rangle$. Under the VMD hypothesis [14], we obtain

$$\langle\gamma(q)\gamma(k)|H_w^{BSW}|K_L^0(p)\rangle = \sum_{V,V'}\left[\frac{X_V(q^2)}{m_V^2}\right]\left[\frac{X_{V'}(k^2)}{m_{V'}^2}\right]\langle V(q)V'(k)|H_w^{BSW}|K_L^0(p)\rangle, \qquad (44)$$

where $V$ and $V'$ denote vector mesons ($\rho^0$, $\omega$ and $\phi$) which couple to the photon. $X_V$ and $X_{V'}$ provide the corresponding coupling strengths. The above factorizable amplitude is strongly suppressed because it is a color suppressed one like that for the $K \to \pi^0\pi^0$ decay. Therefore we can neglect safely the factorizable amplitude for the $K_L \to \gamma\gamma$ decay.

Non-factorizable long distance contributions will be dominated by pole amplitudes since contributions of two and more pion intermediate states are suppressed because of the approximate CP invariance and small phase space volume, respectively. However a sum of pseudo scalar meson ($P_i = \pi^0$, $\eta$, $\eta'$) pole amplitudes [37]

$$A_P(K_L \to \gamma\gamma) = \sum_{P_i}\frac{\langle K_L|\tilde{H}_w|P_i\rangle A(P_i \to \gamma\gamma)}{(m_{P_i}^2 - m_K^2)} \qquad (45)$$

with the usual $\eta$-$\eta'$ mixing angle, $\theta_P \simeq -20°$ [27], is not sufficient [38] to reproduce the observed rate [27], $\Gamma(K_L \to \gamma\gamma)_{\text{expt}} = (7.30 \pm 0.33) \times 10^{-12}$ eV. Therefore we have to take into account some other contributions. Although a possible role of the pseudo scalar glue-ball ($\iota$) through the penguin effect has been considered in Ref. [37], it will be not very important because of its high mass and small rate [27], $\Gamma(\iota \to \gamma\gamma)_{\text{expt}} < 1.2$ keV. Another possible contribution to the $K_L \to \gamma\gamma$ will be the $K^*$ meson pole with the VMD. However there have been some arguments against it [39]. These arguments are based on the field algebra [40] and their weak Hamiltonian consists of *symmetric* products of left-handed currents and transforms like $\mathbf{8_s}$ (not $\mathbf{8_a}$) of $SU_f(3)$. It is much different from the standard model reviewed in Sec. II. Furthermore, the field algebra predicts a finite c-number coefficient [14] of the Schwinger term [41]. It implies [42] that the cross section, $\sigma(e^+e^- \to$ hadrons$)$, goes to zero faster than $s^{-2}$ as $s \to \infty$, where $s$ is the total energy in the center of mass system. However it is inconsistent with experiment [27]. Therefore we should not be restricted by such arguments. Since the VMD in the electro-magnetic interactions of hadrons can be derived [43] independently of the field algebra, we now can be free from the arguments in Ref. [39] even if we assume the VMD. In this way, we can safely take into account the $K^*$ pole contribution in the $K_L \to \gamma\gamma$ decay [25]. Its off-mass-shell amplitude is given by

$$A_{K^*}(K_L \to \gamma\gamma^*(k^2)) = \sum_{V_i}\sum_{V_j}\sqrt{2}X_{V_i}X_{V_j}G_{K^0K^{*0}V_i}\langle K^{*0}|\tilde{H}_w|V_j\rangle_1$$

$$\times\left\{\frac{1}{m_{V_i}^2(m_{K^*}^2 - k^2)(m_{V_j}^2 - k^2)} + \frac{1}{(m_{V_i}^2 - k^2)m_{K^*}^2 m_{V_j}^2}\right\} \qquad (46)$$

with $V_i = \rho^0$, $\omega$ and $\phi$. $X_{V_i}$ is the photon-vector meson coupling strength. The subscript 1 of the matrix element, $\langle K^{*0}|\tilde{H}_w|V_j\rangle_1$, denotes the helicity $|\lambda|$ of the vector meson states

sandwiching $\tilde{H}_w$ as in Eq.(23). The $K^*$ pole amplitude for the $K_L \to \gamma\gamma$ decay is simply obtained by putting $k^2 = 0$ in the above off-mass-shell amplitude, Eq.(46).

The pseudo scalar meson pole amplitude, Eq.(45), can be extrapolated into the off-mass-shell region in the form,

$$A_P(K_L \to \gamma\gamma^*(k^2)) = \sum_{P_i} \frac{\langle K_L | \tilde{H}_w | P_i \rangle A(P_i \to \gamma\gamma)}{(m_{P_i}^2 - m_K^2)(1 - k^2/\Lambda_{P_i}^2)}. \tag{47}$$

The observed form factors for the $\pi^0$, $\eta$ and $\eta' \to \gamma\gamma^*$ decays are approximately described in the form [44], $\sim (1 - k^2/\Lambda_P^2)^{-1}$ with $\Lambda_P \simeq m_\rho$. For more precise arguments, however, we may have to use a more improved result from recent measurements [45], $\Lambda_\pi = 776 \pm 10 \pm 12 \pm 16$ MeV, $\Lambda_\eta = 774 \pm 11 \pm 16 \pm 22$ MeV and $\Lambda_{\eta'} = 859 \pm 9 \pm 18 \pm 20$ MeV. In this way, the amplitude and the form factor for the $K_L \to \gamma\gamma^*$ are approximately given by

$$A(K_L \to \gamma\gamma^*(k^2)) \simeq A_P(K_L \to \gamma\gamma^*(k^2)) + A_{K^*}(K_L \to \gamma\gamma^*(k^2)) \tag{48}$$

and

$$f(k^2) = \frac{A(K_L \to \gamma\gamma^*(k^2))}{A(K_L \to \gamma\gamma)}, \tag{49}$$

respectively, where

$$A(K_L \to \gamma\gamma) = A(K_L \to \gamma\gamma^*(k^2 = 0)). \tag{50}$$

As seen in Eqs.(48) and (49) with Eqs.(45), (46) and (47), the amplitude $A(K_L \to \gamma\gamma^*)$ and the form factor $f(k^2)$ for the $K_L \to \gamma\gamma^*$ have been written approximately in terms of (asymptotic) ground-state-meson matrix elements of $\tilde{H}_w$.

Since the Dalitz decay of $K_L$ proceeds dominantly as $K_L \to \gamma\gamma^* \to \gamma\ell^+\ell^-$, its branching fraction is given by the following formula [46],

$$R_{\gamma\ell^+\ell^-} = \frac{\Gamma(K_L \to \gamma\ell^+\ell^-)}{\Gamma(K_L \to \gamma\gamma)} = [\Gamma(K_L \to \gamma\gamma)]^{-1} \int_{x_{\min}}^1 dx \left[ \frac{d\Gamma(K_L \to \gamma\ell^+\ell^-)}{dx} \right] \tag{51}$$

with $x = k^2/m_K^2$, where

$$[\Gamma(K_L \to \gamma\gamma)]^{-1} \frac{d\Gamma(K_L \to \gamma\ell^+\ell^-)}{dx}$$
$$= \left( \frac{2\alpha}{3\pi} \right) \frac{(1-x)^3}{x} \left[ 1 + 2 \left( \frac{m_\ell}{m_K} \right)^2 \frac{1}{x} \right] \left[ 1 - 4 \left( \frac{m_\ell}{m_K} \right)^2 \frac{1}{x} \right]^{1/2} |f(x)|^2. \tag{52}$$

## VI. COMPARISON WITH EXPERIMENTS

Inserting the parameterization of the asymptotic matrix elements of $\tilde{H}_w$ (in Sec. II and Appendix A) into the general form of the non-factorizable long distance amplitudes given in the previous sections, we can obtain explicit expressions of the amplitudes for the $K \to \pi\pi$ and $K_L \to \gamma\gamma^{(*)}$ decays in addition to the pole contribution, $(\Delta m_K)_{\text{pole}}$, to the $K_L$-$K_S$ mass

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Table II. Branching ratios (%) for the $K \to \pi\pi$ decays. $B_{\mathrm{FA}}$, $B_{\mathrm{NF}}$ and $B_{\mathrm{TOT}}$ are given by the amplitudes $M_{\mathrm{FA}}$, $M_{\mathrm{NF}}$ and $M_{\mathrm{TOT}} = M_{\mathrm{FA}} + M_{\mathrm{NF}}$, respectively. Values of unknown parameters involved in $M_{\mathrm{NF}}$ are taken to be $\delta_0 = 52°$, $k_a = 0.128$, $k_s = 0.095$ and $|\langle \pi^+ | \tilde{H}_w | K^+ \rangle| = 1.99 \times 10^{-7} m_K^2$. The observed lifetimes [27], $\tau(K^\pm)_{\mathrm{expt}} = (1.2386 \pm 0.0024) \times 10^{-8}\,s$ and $\tau(K_S^0)_{\mathrm{expt}} = (0.8934 \pm 0.0008) \times 10^{-10}\,s$, have been used as input data.

| Decay | $B_{\mathrm{FA}}$ | $B_{\mathrm{NF}}$ | $B_{\mathrm{TOT}}$ | Experiment |
|---|---|---|---|---|
| $K_S \to \pi^+ \pi^-$ | 3.2 | 72 | 62 | $68.61 \pm 0.28$ |
| $K_S \to \pi^0 \pi^0$ | 0.05 | 36 | 35 | $31.39 \pm 0.28$ |
| $K^+ \to \pi^+ \pi^0$ | 76 | 22 | 20 | $21.16 \pm 0.14$ |

difference which are summarized in Appendix B. However, they still contain many parameters whose values have not been specified. Before we compare our result with the measurements, we estimate values of these parameters using various experimental data.

Sizes of the amplitudes $A(P_i \to \gamma\gamma)$'s in Eq.(45) can be estimated from the measured decay rates [27], $\Gamma(\pi^0 \to \gamma\gamma)_{\mathrm{expt}} = (7.7 \pm 0.6)$ eV, $\Gamma(\eta \to \gamma\gamma)_{\mathrm{expt}} = (0.46 \pm 0.04)$ keV and $\Gamma(\eta' \to \gamma\gamma)_{\mathrm{expt}} = (4.26 \pm 0.19)$ keV. Their signs are taken to be compatible with the quark model. The $V$-$V'$-$P$, ($V$, $V' = K^*$, $\rho$, $\omega$ and $\phi$; $P = K$, $\pi$, $\eta$ and $\eta'$), coupling constants can be estimated from the observed rates for the radiative decays of $K^*$ by using $SU_f(3)$ symmetry and the VMD with the $\gamma$-$V$ coupling strengths [47], $X_\rho(0) = 0.033 \pm 0.003$ (GeV)$^2$, $X_\omega(0) = 0.011 \pm 0.001$ (GeV)$^2$ and $X_\phi(0) = -0.018 \pm 0.001$ (GeV)$^2$, estimated from data on photo-productions of vector mesons. Although these $\gamma$-$V$ coupling strengths can have momentum square ($k^2$) dependence, we neglect it in this paper since it is mild in the region $k^2 < m_K^2$. From $\Gamma(K^{*0} \to K^0 \gamma)_{\mathrm{expt}} = (0.115 \pm 0.012)$ MeV [27], we obtain $|G_{K^{*0} K^0 \rho^0}| \simeq 0.856$ (GeV)$^{-1}$ and then $G_{\omega \pi^0 \rho^0} = -2 G_{K^{*0} K^0 \rho^0}$ using $SU_f(3)$. In this way, we can reproduce well the observed rate, $\Gamma(\pi^0 \to \gamma\gamma)_{\mathrm{expt}}$.

Size of $\langle \pi | \tilde{H}_w | K \rangle$ can be estimated phenomenologically by fitting the rates for the $K \to \pi\pi$ decays obtained from total amplitudes (sums of the factorized ones in Table I and the non-factorizable ones in Appendix B) to the observed rates. However the latter amplitudes still involve many unknown parameters, i.e., masses and widths of four-quark $[qq][\bar{q}\bar{q}]$ and $(qq)(\bar{q}\bar{q})$ mesons, iso-singlet $S$-wave $\pi\pi$ phase shift $\delta_0$ at $m_K$, the asymptotic ground-state-meson matrix element of $\tilde{H}_w$, $\langle \pi | \tilde{H}_w | K \rangle$, and parameters $k_a$ and $k_s$ providing residues of four-quark $[qq][\bar{q}\bar{q}]$ and $(qq)(\bar{q}\bar{q})$ meson poles as defined in Appendix A. We here take the predicted values of four-quark meson masses. For their widths, we take $\simeq 0.3$ GeV tentatively since they are expected to be considerably broad. For the $S$-wave $\pi\pi$ phase shift, we take $\delta_0 \simeq (50-60)°$ at $m_K$ [48]. Since we do not know, at the present stage, how to estimate values of $\langle \pi | \tilde{H}_w | K \rangle$, $k_a$ and $k_s$, we treat them as adjustable parameters and look for their values to reproduce the observed rates for the $K \to \pi\pi$ decays. We expect that values of $k_a$ and $k_s$ will be much smaller than unity since wave function overlapping between the four-quark and the ground-state $\{q\bar{q}\}_0$ meson states is expected to be small and that the matrix element, for instance, $|\langle \pi^+ | \tilde{H}_w | K^+ \rangle|$, will be much larger, because of soft gluon effects, than the factorized $|\langle \pi^+ | H_w^{BSW} | K^+ \rangle_{\mathrm{FA}}| \simeq 0.232 m_K^2 \times 10^{-7}$. Taking reasonable values of these parameters around $k_a \simeq 0.12$, $k_s \simeq 0.10$ and $|\langle \pi^+ | \tilde{H}_w | K^+ \rangle| \simeq 2.0 m_K^2 \times 10^{-7}$, we can reproduce fairly well the

observed rates for the $K \to \pi\pi$ decays. In Table II where branching ratios, $B_{\mathrm{FA}}$, $B_{\mathrm{NF}}$ and $B_{\mathrm{TOT}}$, are given by the amplitudes, $M_{\mathrm{FA}}$, $M_{\mathrm{NF}}$ and $M_{\mathrm{TOT}} = M_{\mathrm{FA}} + M_{\mathrm{NF}}$, respectively, we show our typical result which is obtained by taking the following values of parameters, $\delta_0 = 52°$, $k_a = 0.128$, $k_s = 0.095$ and $|\langle \pi^+|\tilde{H}_w|K^+\rangle| = 1.99 m_K^2 \times 10^{-7}$, where the observed lifetimes [27], $\tau(K^\pm)_{\mathrm{expt}} = (1.2386 \pm 0.0024) \times 10^{-8}\, s$ and $\tau(K_S^0)_{\mathrm{expt}} = (0.8934 \pm 0.0008) \times 10^{-10}\, s$, have been used as input data. We will use hereafter the above value of $|\langle \pi^+|\tilde{H}_w|K^+\rangle|$ in this paper. As mentioned before, it is seen that $B_{\mathrm{FA}}$ given by the naive factorization cannot reproduce the $|\Delta \mathbf{I}| = \frac{1}{2}$ rule and that the non-factorizable contributions play an essential role. As the consequence, $B_{\mathrm{TOT}}$ can reproduce the observations within about 10 % errors. (Of course, the experimental errors are much smaller [27].)

For $(\Delta m_K)_{\mathrm{SD}}$, we have obtained $\{(\Delta m_K)_{\mathrm{SD}}\}_{\mathrm{NF}} \propto \langle K^0|\tilde{O}_{\Delta S=2}|\bar{K}^0\rangle = 0$ in Eq.(39) using $[SU_f(3)$ symmetry and] the quark counting in Sec. II, and hence

$$(\Delta m_K)_{\mathrm{SD}} = \{(\Delta m_K)_{\mathrm{SD}}\}_{\mathrm{FA}}. \tag{53}$$

Although the value of the coefficient $c_{\Delta S=2}$ at $m_c$ has been calculated [2], we do not definitely know the value of $B_K$. Furthermore, the calculated value of $c_{\Delta S=2}$ (with the NLO corrections) contains still large ambiguities. Therefore, we will treat $B_K$ as an unknown parameter and study the following three cases, (i) $B_K = 1.0$, (ii) $B_K = 0.75$ and (iii) $B_K = 0.50$, later since the lattice QCD suggests that $B_K \sim 0.6$ [49]. When the above NLO results [33] on $\eta_i$'s are used, top quark contribution to $H_{\Delta S=2}$ is not very important as long as real part of the $K^0$-$\bar{K}^0$ mixing amplitude is concerned. Taking $\eta_1 \simeq 1.38$ [2,33], we obtain

$$\frac{(\Delta m_K)_{\mathrm{SD}}}{\Gamma_{K_S}} \simeq 0.49 B_K. \tag{54}$$

Then the pole contribution which has been calculated in Sec. IV should be compared with

$$\begin{aligned}
\frac{(\Delta m_K)_{\mathrm{pole}}}{\Gamma_{K_S}} &\simeq \left(\frac{\Delta m_K}{\Gamma_{K_S}}\right)_{\mathrm{expt}} - \frac{(\Delta m_K)_{\pi\pi}}{\Gamma_{K_S}} - \frac{(\Delta m_K)_{\mathrm{SD}}}{\Gamma_{K_S}} \\
&\simeq (0.477 \pm 0.002) - (0.22 \pm 0.03) - (0.49 B_K),
\end{aligned} \tag{55}$$

where the value of $(\Delta m_K)_{\pi\pi}/\Gamma_{K_S}$ has been given in Ref. [4] as mentioned before.

The remaining parameters involved in the $(\Delta m_K)_{\mathrm{pole}}$ and $\Gamma(K_L \to \gamma\gamma)$ are

$$\tilde{r}_0 \quad \text{and} \quad \tilde{\alpha}_{K^*} = \frac{\langle \rho^+|\tilde{H}_w|K^{*+}\rangle_1}{\langle \pi^+|\tilde{H}_w|K^+\rangle}, \tag{56}$$

where $\tilde{r}_0$ has been defined in Sec. II as a parameter describing the contribution of the penguin relative to $\tilde{O}_-$ in the asymptotic ground-state-meson matrix elements of $\tilde{H}_w$ with the helicity $\lambda = 0$. We now search for values of $\tilde{r}_0$ and $\tilde{\alpha}_{K^*}$ to reproduce $(\Delta m_K)_{\mathrm{expt}}$ and $\Gamma(K_L \to \gamma\gamma)_{\mathrm{expt}}$ mentioned before. [We have already used $\Gamma(K \to \pi\pi)_{\mathrm{expt}}$'s to estimate the size of $\langle \pi^+|\tilde{H}_w|K^+\rangle$.] Inserting the parameterization of the asymptotic ground-state-meson matrix elements of $\tilde{H}_w$ in Sec. II into $(\Delta m_K)_{\mathrm{pole}}$ in Eq.(43) and $A(K_L \to \gamma\gamma)$ in Eq.(50) and using $|\langle \pi^+|\tilde{H}_w|K^+\rangle| \simeq 1.99 m_K^2 \times 10^{-7}$, we can reproduce the values of $(\Delta m_K)_{\mathrm{pole}}/\Gamma_{K_S}$ in Eq.(55) and $\Gamma(K_L \to \gamma\gamma)_{\mathrm{expt}}$ (i) for $\tilde{r}_0 \simeq 0.535$ and $\tilde{\alpha}_{K^*} \simeq 2.61$ with $B_K = 1.0$, (ii) for $\tilde{r}_0 \simeq 0.635$ and $\tilde{\alpha}_{K^*} \simeq 2.34$ with $B_K = 0.75$, and (iii) for $\tilde{r}_0 \simeq 0.715$ and $\tilde{\alpha}_{K^*} \simeq 2.15$ with

$B_K = 0.50$. All the three cases suggest that $\tilde{r}_0 > |c_5/\tilde{c}_-|$. It means that the ground-state-meson matrix element of $O_5$ is actually enhanced relatively to that of $\tilde{O}_-$. It is compatible with $\tilde{\alpha}_{K^*} > 1$ which means that matrix elements of products of left-handed currents taken between helicity $\pm 1$ states is enhanced relatively to the ones between helicity 0 states.

Insertion of the parameterization of the asymptotic ground-state-meson matrix elements of $\tilde{H}_w$ with the above values of $\tilde{r}_0$ and $\tilde{\alpha}_{K^*}$ into Eq.(49) leads to three different form factors for the Dalitz decays of $K_L$. Since the difference among the three cases is not very significant, however, we show only the result on the form factor for (ii) $B_K = 0.75$, $\tilde{r}_0 \simeq 0.635$ and $\tilde{\alpha}_{K^*} \simeq 2.34$ in Fig. I. For experimental data on the form factor to be compared, three different ones have been known, *i.e.*, two of them are from the $\gamma e^+ e^-$ final states [50,51] and the other is from the $\gamma \mu^+ \mu^-$ [52]. The existing data from different types of the final states are not consistent with each other near the $\gamma \mu^+ \mu^-$ threshold as seen in Fig. I. Our result on the form factor for the $K_L \to \gamma \ell^+ \ell^-$ decays seems to prefer to the data [50,51] from the $K_L \to \gamma e^+ e^-$ decay. Therefore our values of the form factor near the threshold of the $K_L \to \gamma \mu^+ \mu^-$ decay are considerably larger than the data from Ref. [52]. At higher $x = k^2/m_K^2 (> 0.4)$, however, our results are consistent with almost all the data within their
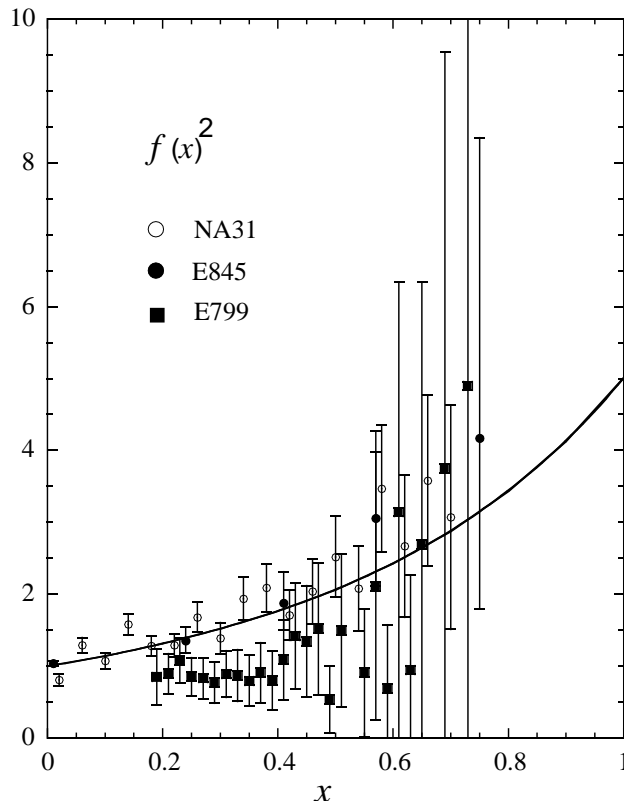


Fig. I. Form factor squared for $B_K = 0.75$, $\tilde{r}_0 = 0.635$ and $\tilde{\alpha}_{K^*} = 2.34$. Circles and bullets are obtained from the $\gamma e^+ e^-$ final states by E845 at BNL [51] and NA31 at CERN [50], respectively, and squares from the $\gamma \mu^+ \mu^-$ final states by E799 at FNAL [52].

Table III. Branching fractions for the Dalitz decays of $K_L$ for (i) $\tilde{r}_0 = 0.535$ and $\tilde{\alpha}_{K^*} = 2.61$ in the case of $B_K = 1.0$, (ii) $\tilde{r}_0 = 0.635$ and $\tilde{\alpha}_{K^*} = 2.34$ in the case of $B_K = 0.75$, and (iii) $\tilde{r}_0 = 0.715$ and $\tilde{\alpha}_{K^*} = 2.15$ in the case of $B_K = 0.50$. The data values with ($*$), ($\dagger$) and ($\ddagger$) are taken from Refs. [27], [52] and [55], respectively.

| | $R_{\gamma e^+ e^-} (\times 10^{-2})$ | $R_{\gamma \mu^+ \mu^-} (\times 10^{-4})$ |
|---|---|---|
| (i)  $B_K = 1.0$ ; $\tilde{r}_0 = 0.535$, $\tilde{\alpha}_{K^*} = 2.61$ | 1.6 | 6.5 |
| (ii)  $B_K = 0.75$; $\tilde{r}_0 = 0.635$, $\tilde{\alpha}_{K^*} = 2.34$ | 1.6 | 6.5 |
| (iii) $B_K = 0.50$; $\tilde{r}_0 = 0.715$, $\tilde{\alpha}_{K^*} = 2.15$ | 1.6 | 6.4 |
| Experiments | $(1.6 \pm 0.1)$ ($*$) | $(5.6 \pm 0.8)$ ($\dagger$) <br> $(5.9 \pm 1.8)$ ($\ddagger$) |

large errors.

Substituting the above results on the form factor for the Dalitz decays of $K_L$ into the formula Eq.(51) with Eq.(52), we can calculate their branching fractions [53] as listed in Table III. The rate for the Dalitz decay of $K_L$ is mainly determined by the values of the form factor near the threshold. Therefore, the rate $\Gamma(K_L \to \gamma e^+ e^-)$ is not very useful to distinguish different theories since its threshold is close to $x = 0$ where the form factor is normalized to be $f(0) = 1$. Since the threshold of the $K_L \to \gamma \mu^+ \mu^-$ decay is considerably distant from the normalization point $x = 0$, however, we may discriminate different models using this decay when they give remarkably different values of the form factor around the $\gamma \mu^+ \mu^-$ threshold. In fact, for example, the BMS model [46] with $\alpha_{K^*} = -0.28$ which provides the best fit to the data on the form factor by NA31 [50] predicts $R_{\gamma \mu^+ \mu^-} \simeq 7.3 \times 10^{-4}$ which is substantially larger than our results and the data from E799 [52]. The existing theoretical analyses [46,54] involving the above BMS model have been restricted by the argument in Ref. [39] which is based on the field algebra and therefore, in these theories, the $K^*$ pole amplitude has to vanish in the $K_L \to \gamma \gamma$ while it survives in the Dalitz decays of $K_L$. In this case, however, it will be hard [38] to reproduce the observed $\Gamma(K_L \to \gamma \gamma)$ in consistency with the $K \to \pi \pi$ decays if the usual $\eta$-$\eta'$ mixing angle $\theta_P \simeq -20°$ is taken. Additionally, they involve implicitly serious problems mentioned before.

As seen in Table III, our three results on $R_{\gamma \mu^+ \mu^-}$ for the above three different values of $B_K$ are not very different from each other since the corresponding form factors are not very different in the three cases and are a little larger than the data from Refs. [52] but still consistent with the data from Ref. [55]. It is because our values of the form factor prefer to the data from the $K_L \to \gamma e^+ e^-$ decay [50,51] but are considerably larger, around the $\gamma \mu^+ \mu^-$ threshold, than the data by E799 [52] from the $K_L \to \gamma \mu^+ \mu^-$ decay.

## VII. SUMMARY

We have investigated $K \to \pi \pi$, $\Delta m_K$, $K_L \to \gamma \gamma$ and the Dalitz decays of $K_L$ systematically. Our result has included many parameters. Although their values have been estimated by using existing experimental data, three ($B_K$, $\tilde{r}_0$ and $\tilde{\alpha}_{K^*}$) of them have still remained

unknown. Therefore, in the three cases, (i) $B_K = 1.0$ in which $\Delta m_K / \Gamma_{K_S}$ is almost saturated by the short distance factorized contribution, (ii) $B_K = 0.75$ and (iii) $B_K = 0.50$ in which about half of the observed value of $\Delta m_K / \Gamma_{K_S}$ is occupied by the long distance non-factorizable contribution, we have searched for possible values of $\tilde{r}_0$ and $\tilde{\alpha}_{K^*}$ which reproduce the existing *data* on $\Delta m_K / \Gamma_{K_S}$ and $\Gamma(K_L \to \gamma\gamma)$ simultaneously. Then, using the results, we have calculated the form factor for the Dalitz decays of $K_L$ and their decay rates. The three results on the form factor have been not very different from each other. On the other hand, the existing data on the form factor from the $K_L \to \gamma e^+ e^-$ decay [50,51] and the $K_L \to \gamma \mu^+ \mu^-$ decay [52] are not consistent with each other near the $\gamma\mu^+\mu^-$ threshold. Our results on the form factor around the threshold of the $K_L \to \gamma\mu^+\mu^-$ decay seem to be consistent with the data from the $\gamma e^+ e^-$ final state but a little larger than the data from the $\gamma\mu^+\mu^-$ final state. The rate for the Dalitz decay is controlled dominantly by the value of its form factor near the threshold so that $\Gamma(K_L \to \gamma\mu^+\mu^-)$ will be useful to distinguish different models in contrast with $\Gamma(K_L \to \gamma e^+ e^-)$. However, more theoretical and experimental investigations of the Dalitz decays of $K_L$ will be needed, since their ambiguities are still large.

# APPENDIX A: ASYMPTOTIC MATRIX ELEMENTS OF $\tilde{H}_W$

Constraints on asymptotic matrix elements of non-factorizable four-quark operators $\tilde{O}_{\pm}$ have been given in Eqs.(16), (18) and (19) in the text by counting all possible connected quark-line diagrams. We here present explicitly the results in the strangeness changing case since we have already presented charm changing ones in Ref. [21]. (Notations of the four-quark mesons have also given in Ref. [21].)

(i) Asymptotic matrix elements of $\tilde{H}_w$ between ground-state $\{q\bar{q}\}_0$ meson states:
   Relations among asymptotic ground-state-meson matrix elements of $\tilde{O}_-$,

$$\begin{cases} \langle\pi^+|\tilde{O}_-|K^+\rangle = -\sqrt{2}\langle\pi^0|\tilde{O}_-|K^0\rangle = -\sqrt{2}\langle\eta_0|\tilde{O}_-|K^0\rangle, \\ \langle\eta_s|\tilde{O}_-|K^0\rangle = 0, \end{cases} \tag{A1}$$

and their vector meson counterparts ($K$, $\pi$, $\eta_0$ and $\eta_s \to K^*$, $\rho$, $\omega$ and $\phi$) which are compatible with the constraint, Eq.(16), and the constraints on the asymptotic matrix elements of $O_5$, Eq.(20), in Sec. II lead to the $|\Delta\mathbf{I}| = \frac{1}{2}$ rule for the asymptotic ground-state-meson matrix elements of $\tilde{H}_w$,

$$\langle\pi^+|\tilde{H}_w|K^+\rangle = -\sqrt{2}\langle\pi^0|\tilde{H}_w|K^0\rangle, \quad \langle\rho^+|\tilde{H}_w|K^{*+}\rangle = -\sqrt{2}\langle\rho^0|\tilde{H}_w|K^{*0}\rangle, \tag{A2}$$

and the parameterizations, Eqs.(22) and (23), in Sec. II.

(ii) Asymptotic matrix elements of $\tilde{H}_w$ between $\{q\bar{q}\}_0$ and $[qq][\bar{q}\bar{q}]$ meson states:
   Relations among asymptotic matrix elements of $\tilde{O}_-$ taken between the ground-state $\{q\bar{q}\}_0$ and $[qq][\bar{q}\bar{q}]$ meson states,

$$\begin{cases} \langle\pi^+|\tilde{O}_-|\hat{\kappa}^+\rangle = -\sqrt{2}\langle\pi^0|\tilde{O}_-|\hat{\kappa}^0\rangle = \langle\sigma|\tilde{O}_-|K^0\rangle = \cdots, \\ \langle\sigma^s|\tilde{O}_-|K^0\rangle = 0, \end{cases} \tag{A3}$$

which are compatible with the constraint of Eq.(18) lead to

$$\langle\pi^+|\tilde{H}_w|\hat{\kappa}^+\rangle = -\sqrt{2}\langle\pi^0|\tilde{H}_w|\hat{\kappa}^0\rangle = \langle\sigma|\tilde{H}_w|K^0\rangle = \cdots = \frac{k_a}{2A_a}\langle\pi^+|\tilde{H}_w|K^+\rangle, \tag{A4}$$

where $A_a$ is the invariant matrix element of axial charge defined by $A_a = -\frac{1}{2}\langle\hat{\kappa}^+|A_{\pi^+}|K^0\rangle$. In Eq.(A4), we have parameterized the matrix elements using the ground-state-meson matrix element of $\tilde{H}_w$ and a parameter $k_a$ introduced. In the above, asymptotic $SU_f(3)$ symmetry (or nonet symmetry with respect to asymptotic matrix elements of charges) has been assumed.

(iii) Asymptotic matrix elements of $\tilde{H}_w$ between $\{q\bar{q}\}_0$ and $(qq)(\bar{q}\bar{q})$ meson states:
   Relations among asymptotic matrix elements of $\tilde{O}_+$,

$$\sqrt{\frac{3}{2}}\langle E_{\pi\pi}^+|\tilde{O}_+|K^+\rangle = -\frac{3}{2}\langle E_{\pi\pi}^0|\tilde{O}_+|K^0\rangle = \frac{3}{\sqrt{2}}\langle C^0|\tilde{O}_+|K^0\rangle = -\frac{3}{4}\langle\pi^+|\tilde{O}_+|E_{\pi K}^+\rangle$$

$$= -\frac{3}{\sqrt{2}}\langle\pi^0|\tilde{O}_+|E_{\pi K}^0\rangle = \frac{\sqrt{3}}{2}\langle\pi^-|\tilde{O}_+|E_{\pi K}^-\rangle = \frac{3}{\sqrt{2}}\langle\pi^+|\tilde{O}_+|C_K^+\rangle = 3\langle\pi^0|\tilde{O}_+|C_K^0\rangle = \cdots, \tag{A5}$$

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which are compatible with the constraint of Eq.(19) lead to

$$\sqrt{\frac{3}{2}}\langle E_{\pi\pi}^+|\tilde{H}_w|K^+\rangle = -\frac{3}{2}\langle E_{\pi\pi}^0|\tilde{H}_w|K^0\rangle = \frac{3}{\sqrt{2}}\langle C^0|\tilde{H}_w|K^0\rangle = -\frac{3}{4}\langle \pi^+|\tilde{H}_w|E_{\pi K}^+\rangle$$

$$= -\frac{3}{\sqrt{2}}\langle \pi^0|\tilde{H}_w|E_{\pi K}^0\rangle = \frac{\sqrt{3}}{2}\langle \pi^-|\tilde{H}_w|E_{\pi K}^-\rangle = \frac{3}{\sqrt{2}}\langle \pi^+|\tilde{H}_w|C_K^+\rangle = 3\langle \pi^0|\tilde{H}_w|C_K^0\rangle = \cdots$$

$$= \frac{k_s}{A_s}\langle \pi^0|\tilde{H}_w|K^0\rangle, \qquad \text{(A6)}$$

where $A_s$ is the invariant matrix element of axial charge defined by $A_s = \langle C_K^+|A_{\pi^+}|K^0\rangle$. The above equations imply that the asymptotic matrix elements of $\tilde{H}_w$ between $\{q\bar{q}\}_0$ and $(qq)(\bar{q}\bar{q})$ meson states can violate the $|\Delta \mathbf{I}| = \frac{1}{2}$ rule. In Eq.(A6), we have parameterized the matrix elements using the ground-state-meson matrix element of $\tilde{H}_w$ and a parameter $k_s$ introduced. In the above, asymptotic $SU_f(3)$ symmetry (or nonet symmetry with respect to asymptotic matrix elements of charges) has been assumed.

## APPENDIX B: NON-FACTORIZABLE AMPLITUDES

We here list approximate non-factorizable amplitudes for the $K \to \pi\pi$ decays, pole contribution to the $K_L$-$K_S$ mass difference and long distance amplitude for the $K_L \to \gamma\gamma^{(*)}$.

(i) Non-factorizable amplitudes for the $K \to \pi\pi$ decays:

Inserting the constraints on asymptotic matrix elements of $\tilde{H}_w$ in Appendix A into Eq.(26) with Eqs.(27) and (28), we obtain the following non-factorizable amplitudes for the $K \to \pi\pi$ decays,

$$M_{\text{NF}}(K_S^0 \to \pi^+\pi^-)$$
$$\simeq -\frac{i}{f_\pi}\langle \pi^+|\tilde{H}_w|K^+\rangle\left\{ e^{i\delta_0(\pi\pi)} - \left[2\left(\frac{m_K^2 - m_\pi^2}{m_{\hat{\sigma}}^2 - m_K^2}\right) + \left(\frac{m_K^2 - m_\pi^2}{m_{\hat{\kappa}}^2 - m_\pi^2}\right)\right]k_a \right.$$
$$\left. + \left[2\left(\frac{m_K^2 - m_\pi^2}{m_{E_{\pi\pi}}^2 - m_K^2}\right) + \left(\frac{m_K^2 - m_\pi^2}{m_{E_{\pi K}}^2 - m_\pi^2}\right)\right]k_s \right\}, \qquad \text{(B1)}$$

$$M_{\text{NF}}(K_S^0 \to \pi^0\pi^0)$$
$$\simeq -\frac{i}{f_\pi}\langle \pi^+|\tilde{H}_w|K^+\rangle\sqrt{\frac{1}{2}}\left\{ e^{i\delta_0(\pi\pi)} - \left[2\left(\frac{m_K^2 - m_\pi^2}{m_{\hat{\sigma}}^2 - m_K^2}\right) + \left(\frac{m_K^2 - m_\pi^2}{m_{\hat{\kappa}}^2 - m_\pi^2}\right)\right]k_a \right.$$
$$\left. - \left[2\left(\frac{m_K^2 - m_\pi^2}{m_{E_{\pi\pi}}^2 - m_K^2}\right) + \left(\frac{m_K^2 - m_\pi^2}{m_{E_{\pi K}}^2 - m_\pi^2}\right)\right]k_s \right\}, \qquad \text{(B2)}$$

$$M_{\text{NF}}(K^+ \to \pi^+\pi^0) \simeq -\frac{i}{f_\pi}\langle \pi^+|\tilde{H}_w|K^+\rangle\left\{ \left[2\left(\frac{m_K^2 - m_\pi^2}{m_{E_{\pi\pi}}^2 - m_K^2}\right) + \left(\frac{m_K^2 - m_\pi^2}{m_{E_{\pi K}}^2 - m_\pi^2}\right)\right]k_s \right\}, \qquad \text{(B3)}$$

where pole contributions of vector mesons, orbitally and radially excited mesons have been neglected and the mass relations [22], $m_{E_{\pi\pi}} = m_C$ and $m_{E_{\pi K}} = m_{C_K}$, have been used.

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(ii) Pole contribution to the $K_L$-$K_S$ mass difference:

Inserting the parameterization of the asymptotic ground-state-meson matrix elements of $\tilde{H}_w$ in Eqs.(22) and (23) into Eq.(43), we obtain the following pole contribution to $\Delta m_K$,

$$\frac{(\Delta M_K)_{\text{pole}}}{\Gamma_{K_S}} \simeq \frac{|\langle\pi^0|\tilde{H}_w|\bar{K}^0\rangle|^2}{m_K\Gamma_{K_S}(m_K^2 - m_\pi^2)}\left\{1 + \left(\frac{m_K^2 - m_\pi^2}{m_K^2 - m_\eta^2}\right)P_\eta^2 + \left(\frac{m_K^2 - m_\pi^2}{m_K^2 - m_{\eta'}^2}\right)P_{\eta'}^2\right\}, \tag{B4}$$

where

$$P_\eta = \left(\frac{1 - \tilde{r}_0}{1 + \tilde{r}_0}\right)a_\eta^0 - \left(\frac{\sqrt{2}\tilde{r}_0}{1 + \tilde{r}_0}\right)a_\eta^s \quad \text{and} \quad P_{\eta'} = \left(\frac{1 - \tilde{r}_0}{1 + \tilde{r}_0}\right)a_{\eta'}^0 - \left(\frac{\sqrt{2}\tilde{r}_0}{1 + \tilde{r}_0}\right)a_{\eta'}^s. \tag{B5}$$

$a_i^0$ and $a_i^s$, $(i = \eta$ and $\eta')$, are the $\eta$-$\eta'$ mixing parameters whose explicit expression is given in Eqs.(21) in the text.

(iii) Non-factorizable long distance amplitude for the $K_L \to \gamma\gamma^*$ decay:

Inserting the parameterization of the asymptotic ground-state-meson matrix elements of $\tilde{H}_w$ in Eqs.(22) and (23) into Eq.(48) with Eqs.(46) and (47) in the text, we obtain the following long distance amplitude for the $K_L \to \gamma\gamma^*$ decay,

$$A(K_L \to \gamma\gamma^*(k^2)) \simeq A_P(K_L \to \gamma\gamma^*(k^2)) + A_{K^*}(K_L \to \gamma\gamma^*(k^2)), \tag{B6}$$

where

$$A_P(K_L \to \gamma\gamma^*) \simeq \frac{\langle\pi^0|\tilde{H}_w|\bar{K}_L^0\rangle}{(m_K^2 - m_\pi^2)} \times \left\{\frac{A(\pi^0 \to \gamma\gamma)}{1 - \left(\frac{m_K}{\Lambda_{\pi^0}}\right)^2 x}\right.$$

$$+ \left[\frac{A(\eta \to \gamma\gamma)}{1 - \left(\frac{m_K}{\Lambda_\eta}\right)^2 x}\right]\left(\frac{m_K^2 - m_\pi^2}{m_K^2 - m_\eta^2}\right)P_\eta + \left[\frac{A(\eta' \to \gamma\gamma)}{1 - \left(\frac{m_K}{\Lambda_{\eta'}}\right)^2 x}\right]\left(\frac{m_K^2 - m_\pi^2}{m_K^2 - m_{\eta'}^2}\right)P_{\eta'}\right\} \tag{B7}$$

and

$$A_V(K_L \to \gamma\gamma^*) \simeq \sqrt{2}\frac{\langle\rho^0|\tilde{H}_w|\bar{K}^{*0}\rangle G_{K^{*0}K^0\rho^0}}{(m_{K^*}^2 - k^2)(m_\rho^2 - k^2)}\left[\frac{X_\rho(0)}{m_\rho^2}\right]X_\rho(k^2)F_{KK^*V}(0)F_{K^*V}(k^2)$$

$$+ \sqrt{2}\frac{\langle\rho^0|\tilde{H}_w|\bar{K}^{*0}\rangle G_{K^{*0}K^0\rho^0}}{m_{K^*}^2(m_\rho^2 - k^2)}\left[\frac{X_\rho(0)}{m_\rho^2}\right]X_\rho(k^2)F_{KK^*V}(k^2)F_{K^*V}(0) \tag{B8}$$

with

$$F_{KK^*V}(k^2) = 1 + \frac{G_{K^{*0}K^0\omega}}{G_{K^{*0}K^0\rho^0}}\frac{X_\omega(k^2)}{X_\rho(k^2)}\left(\frac{m_\rho^2 - k^2}{m_\omega^2 - k^2}\right) + \frac{G_{K^{*0}K^0\phi}}{G_{K^{*0}K^0\rho^0}}\frac{X_\phi(k^2)}{X_\rho(k^2)}\left(\frac{m_\rho^2 - k^2}{m_\phi^2 - k^2}\right) \tag{B9}$$

and

$$F_{K^*V}(k^2) = 1 + \left(\frac{1 - \tilde{r}_1}{1 + \tilde{r}_1}\right)\frac{X_\omega(k^2)}{X_\rho(k^2)}\left(\frac{m_\rho^2 - k^2}{m_\omega^2 - k^2}\right) - \left(\frac{\sqrt{2}\tilde{r}_1}{1 + \tilde{r}_1}\right)\frac{X_\phi(k^2)}{X_\rho(k^2)}\left(\frac{m_\rho^2 - k^2}{m_\phi^2 - k^2}\right). \tag{B10}$$

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